7.7 Fundamental Matrices

Consider

$$\vec{x}' = P(t)\vec{x}. \qquad (1)$$

Suppose that

$$\{\vec{x}^{(1)}, \vec{x}^{(2)}, \cdots, \vec{x}^{(n)}\}$$

is a fundamental set of solutions to (1), and denote

$$\vec{x}^{(i)} = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}, \quad \text{i.e. } x_{ji} \triangleq x_j^{(i)}.$$

Set

$$\Psi(t) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = (\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}).$$

Then $\Psi(t)$ is called a fundamental matrix for system (1).

Ex 1. Find a fundamental matrix for system

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}.$$

Sol. We need find two linearly independent solutions.
The characteristic equation is

$$\det(A - rI) = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (r-3)(r+1) = 0,$$

thus the eigenvalues are $r_1 = 3$, $r_2 = -1$. Solving the eigenvalue problem

$$(A - r_i I)\vec{\xi}^{(i)} = 0, \quad i = 0, 1,$$

one obtains

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore, we obtain two linearly independent s/n

$$\vec{x}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \vec{x}^{(2)} = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

The fundamental matrix Ψ can be chosen

$$\Psi(t) = \left(\vec{x}^{(1)}, \vec{x}^{(2)}\right) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$

Proposition 1 Assume that $P(t)$ is continuous on an interval I. Let $\Psi(t)$ be a fundamental matrix for system (1) on I. Then

(i) $\det \Psi(t) \neq 0$, $t \in I$;

(ii) $\frac{d}{dt}\Psi(t) = P(t)\Psi(t)$, $t \in I$;

(iii) $\frac{d}{dt}\det\Psi(t) = \operatorname{tr}P(t)\det\Psi(t)$, $t \in I$;

(iv) $\Psi(t)M$ is still a fundamental matrix, for any nonsingular matrix M, in particular

$$\Psi(t)\Psi(t_0)^{-1} \triangleq \Phi(t)$$

is a fundamental matrix for system (1), with

$$\Phi(t_0) = I;$$

(v) the general solution can be expressed as

$$\vec{x} = \Psi(t)\vec{c}, \quad \vec{c} \in \mathbb{R}^n;$$

(vi) the unique solution to

$$\begin{cases} \frac{d}{dt}\vec{x} = P(t)\vec{x}, & t \in I, \\ \vec{x}(t_0) = \vec{x_0} \end{cases}$$

is given by $\vec{x} = \Psi(t)\Psi(t_0)^{-1}\vec{x_0}$.

<u>Proposition 2</u> (the inverse of fundamental matrix)

Let $\Psi(t)$ be a fundamental matrix for system (1) on I. Then $\Psi^{-1}(t)$, the matrix inverse of $\Psi(t)$, satisfies

$$\frac{d}{dt}\Psi^{-1}(t) = -\Psi^{-1}(t)P(t),$$

for $t \in I$.

<u>Proof</u>. Denote $\Phi(t) = \Psi^{-1}(t)$. Then

$$I = \Psi(t)\Phi(t), \quad t \in I.$$

Differentiating the above identity w.r.t. t yields

$$0 = \frac{d}{dt}\Psi(t)\,\Phi(t) + \Psi(t)\frac{d}{dt}\Phi(t)$$
$$= P(t)\Psi(t)\Phi(t) + \Psi(t)\frac{d}{dt}\Phi(t)$$
$$= P(t) + \Psi(t)\frac{d}{dt}\Phi(t), \quad t \in I.$$

Multiplying both sides of the above equation by the matrix $\Phi(t)$ on the left, and noticing that $\Phi\Psi = I$, one obtains the conclusion. #

• The exponential function of matrices e^A.

Let A be a constant matrix. Consider

$$\Phi(t) = e^{At} \triangleq I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots$$

One can easily show the convergence of the series

$$I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots$$

and thus e^{At} is well-defined. Moreover

$$\Phi'(t) = A + tA^2 + \frac{t^2}{2!}A^3 + \cdots + \frac{t^{n-1}}{(n-1)!}A^n$$

$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^{n-1}}{(n-1)!}A^{n-1} + \cdots\right)$$

$$= A\,\Phi(t).$$

Note that $\Phi(0) = I$. Therefore

$$\Phi(t) = e^{At}$$

is a fundamental matrix for system

$$\vec{x}' = A\vec{x}, \qquad (2)$$

and

$$\vec{x} = e^{At}\vec{x}_0$$

is the unique solution to (2) with initial value $\vec{x}_0$.

Proposition 3

(i) $e^{O_{n\times n}} = I$;

(ii) If $AB=BA$, then

$$e^{A+B} = e^A e^B;$$

(iii) $(e^A)^{-1} = e^{-A}$;

(iv) $e^{PAP^{-1}} = Pe^A P^{-1}$.

Proof (i) & (iv) follow directly from the definition, and direct calculations; (iii) follows from (i) & (ii). We only prove (ii): Since $AB=BA$, we have

$$(A+B)^n = \sum_{i=0}^{n} C_n^i A^i B^{n-i},$$

and thus

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} C_n^i A^i B^{n-i}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1}{i!}\frac{1}{(n-i)!} A^i B^{n-i} = \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{A^i}{i!}\frac{B^j}{j!}$$

$$= \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) = e^A e^B.$$

For $A_{n\times n}=(a_{ij})_{n\times n}$, we define

$$\|A\| = \max_{1\le i,j\le n} |a_{ij}|.$$

Then it's easy to check

$$\|AB\| \le n\|A\|\|B\|, \text{ for } A_{n\times n} \text{ and } B_{n\times n},$$

and thus

$$\|A_1A_2\cdots A_k\| \le n^{k-1}\|A_1\|\|A_2\|\cdots\|A_k\|. \tag{3}$$

Note that

$$\begin{aligned}
A^k - B^k &= A^k + \sum_{i=1}^{k-1} A^{k-i}B^i - \sum_{j=1}^{k-1} A^{k-j}B^j - B^k \\
&= \sum_{i=0}^{k-1} A^{k-i}B^i - \sum_{j=1}^{k} A^{k-j}B^j \\
&= \sum_{j=1}^{k} A^{k-j+1}B^{j-1} - \sum_{j=1}^{k} A^{k-j}B^j \\
&= \sum_{j=1}^{k} \left(A^{k-j}AB^{j-1} - A^{k-j}BB^{j-1}\right) \\
&= \sum_{j=1}^{k} A^{k-j}(A-B)B^{j-1}.
\end{aligned} \tag{4}$$

Using (2) and (3), we deduce

$$\|A^k - B^k\| \leq \sum_{j=1}^{k} \|A^{k-j}(A-B)B^{j-1}\|$$

$$\leq \sum_{j=1}^{k} n^{k-1} \|A\|^{k-j} \|B\|^{j-1} \|A-B\|$$

$$\leq \sum_{j=1}^{k} \left(n^{k-1} M^{k-1} \|A-B\|\right)$$

$$= k n^{k-1} M^{k-1} \|A-B\|, \qquad (5)$$

where $M = \max\{\|A\|, \|B\|\}$.

<u>Proposition 4</u> (Lipschitz Continuity of e^A w.r.t A)

$$\|e^A - e^B\| \leq \|A-B\| e^{\max\{n\|A\|, n\|B\|\}},$$

for any $A = A_{n\times n}$ and $B = B_{n\times n}$.

<u>Proof</u> By definition of e^A, we have

$$e^A - e^B = \sum_{k=1}^{\infty} \frac{1}{k!}(A^k - B^k).$$

It follows from (4) that

$$\|e^A - e^B\| \leq \sum_{k=1}^{\infty} \frac{1}{k!}\|A^k - B^k\| \leq \sum_{k=1}^{\infty} \frac{(nM)^{k-1}}{(k-1)!} \|A-B\|$$

$$= \sum_{k=0}^{\infty} \frac{(nM)^k}{k!} \|A-B\| = e^{nM} \|A-B\|. \quad \#$$

The Jordan normal form of A

$$A = PJP^{-1}, \tag{6}$$

Where P is nonsingular, and

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}_{n\times n}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{l_i \times l_i}$$

the size l_i ~~of~~ J_i is less than the algebraic multiplicity of λ_i, and some of λ_i may equival to some other.

By (iv) of Proposition 2, we have

$$e^{At} = e^{PJtP^{-1}} = Pe^{Jt}P^{-1}. \tag{7}$$

We need to calculate e^{Jt}. Note that

$$J^n = \begin{bmatrix} J_1^n & & & \\ & J_2^n & & \\ & & \ddots & \\ & & & J_k^n \end{bmatrix},$$

we have

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_k t} \end{bmatrix}. \tag{8}$$

Therefore, we only need to calculate $e^{J_i t}$.

Note that

$$J_i = \lambda_i I + E,$$

$$I = I_{l_i \times l_i}, \quad E = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{l_i \times l_i},$$

$$E^m = \begin{array}{c} \overset{m+1\text{ th}}{\downarrow} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ & \ddots & & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & 1 \\ & & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ & & & & & 0 \end{bmatrix} \end{array}, \quad \text{when } m \leq l_i - 1,$$

$$E^m = 0, \quad \text{when } m \geq l_i,$$

we have

$$e^{J_i t} = e^{(\lambda_i I + E)t} = e^{\lambda_i I t} e^{Et}$$

$$= e^{\lambda_i t}\left(I + tE + \cdots + \frac{t^{l_i - 1}}{(l_i - 1)!} E^{l_i - 1}\right).$$

$$= e^{\lambda_i t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{l_i-1}}{(l_i-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix}_{l_i\times l_i} \qquad (9)$$

Combining (6), (7), (8), (9), one can give the expression of e^{At}

Note that P is nonsingular, e^{At} is a fundamental matrix for system (2), by (iv) of Proposition 7, $\Psi(t) = e^{At}P$ is also a fundamental matrix for system (2). By (7), we have

$$\Psi(t) = Pe^{Jt}.$$

Write P in the block form as

$$P = (P_1, P_2, \cdots, P_k)$$

where P_1 is the first l_1 columns of P, P_2 is the l_1+1-th to l_1+l_2 columns of P, $\cdots$

Then by (5), we have

$$\Psi(t) = Pe^{Jt} = (P_1, P_2, \cdots, P_k)\begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_k t}\end{bmatrix}$$

$$= (P_1 e^{J_1 t}, P_2 e^{J_2 t}, \cdots, P_k e^{J_k t}). \tag{10}$$

By (6) and (7), one can see that each elements of $P_i e^{J_i t}$ have the form

$$q_{l_i - 1}(t)\, e^{\lambda_i t}, \qquad \deg q_{l_i-1}(t) \leq l_i - 1 \,(\leq s_i - 1),$$

where $q_{l_i-1}(t)$ is polynomial of degree less than $l_i - 1$.

<u>Proposition 5</u> Let $A = A_{n\times n}$, $\lambda_1, \lambda_2, \cdots, \lambda_n$ (may repeat) the eigenvalues of A, with multiplicities $s_1, s_2, \cdots, s_n$, respectively. Then system

$$\vec{x}' = A\vec{x}$$

has a fundamental matrix Ψ of the form

$$\Psi(t) = (\vec{q}_1(t) e^{\lambda_1 t}, \cdots, \vec{q}_n(t) e^{\lambda_n t}),$$

where $\vec{q}_i(t)$ is a vector-valued polynomial of degree less than $s_i - 1$.

<u>Exercise</u> Calculate e^{Jt}, where

$$J = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}.$$

<u>Sol.</u>

$$J = \lambda I + E, \quad E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $IE = EI$, and $E^m = 0$, for $m \geq 3$,

$$e^{Jt} = e^{(\lambda I + E)t} = e^{\lambda It} e^{Et}$$

$$= e^{\lambda t}\left(I + tE + \frac{t^2}{2!}E^2\right)$$

$$= e^{\lambda t}\left(I + \begin{pmatrix} 0 & t & \\ & 0 & t \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}\right)$$

$$= e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix}.$$